International Journal of Theoretical Physics, Vol. 32, No. 10, 1993

Pseudoadditive States on a Logic

Radko Mesiar[1]

Received April 30, 1993

We propose an extension of the notion of a state on a logic to a pseudoadditive state on a logic replacing the ordinary addition by a pseudoaddition. As a special class we underline the possibility states based on the operation of supremum. For a Boolean algebra **B** and a logic **L**, we study the extension of the pseudoadditive states on **B** and **L** to a pseudoadditive state on their Pták sum **B** + **L**.

1. INTRODUCTION

In classical measure theory there are several approaches to generalizing the notion of a measure based on replacing the usual addition on the real line by some other "reasonable" binary operation. Let us call this operation a pseudoaddition. If one restricts consideration to the generalizations of a probability measure, it is enough to deal with the pseudoaddition on the unit interval [0, 1]. For the building up of an integration theory, it is necessary to introduce another binary operation, say a pseudomultiplication. The recent results on this topic include Weber (1984), Riečanová (1982), and Sugeno and Murofushi (1987). An interesting special case is the possibility theory based on the supremum $\vee$ instead of the addition $+$ (Zadeh, 1978). Similar ideas in the fuzzy set theory are developed in Klement and Weber (1991) and Butnariu and Klement (1991).

In quantum logic theory, probability measures on a logic represent states of a described physical system and therefore are called *states* on a logic. However, the nature of a physical system need not be additive. This encourages us to introduce the notion of a $\oplus$*-state* (*pseudoadditive state*), where $\oplus$ is a pseudoaddition replacing the ordinary addition $+$. In particular, for $\oplus = {}_\vee$ (i.e., $\oplus$ is the supremum of reals), a $\vee$-state will be called a *possibility state*.

[1]Slovak Technical University Bratislava, 813 68 Bratislava, Slovakia.

Recently Pták (1986) introduced a logic $\mathscr{L}$ containing (as an embedding) a given Boolean algebra **B** and a given logic **L**. $\mathscr{L}$ is now called a Pták sum, $\mathscr{L} = \mathbf{B} + \mathbf{L}$. States on **B** and **L** induce the states on $\mathscr{L}$ in a manner similar to the integration of a simple function in probability theory. An analogous problem for $\oplus$-states is studied in Section 3. The primary task here is to find a suitable pseudomultiplication $\odot$ corresponding to $\oplus$. In general, no convenient $\odot$ corresponding to a given $\oplus$ may exist. For the possibility states, a suitable pseudomultiplication is, e.g., $\wedge$, i.e., the infimum of reals.

2. PSEUDOADDITIVE STATES

Let **L** be a quantum logic (Beltrametti and Cassinelli, 1981; Varadarajan, 1968) (or simply a logic), i.e., **L** is an orthomodular σ-orthocomplete orthoposet, i.e., a partially ordered set which contains the smallest element **0** and the greatest element **1**, on which an orthocomplementation map $\perp: \mathbf{L} \to \mathbf{L}$ is defined so that the following conditions are fulfilled:

(i) $(a^\perp)^\perp = a$ for any $a \in \mathbf{L}$ (idempotency).

(ii) For any $a, b \in \mathbf{L}$, $a \le b$, one has $b^\perp \le a^\perp$ (order reversing).

(iii) The greatest lower bound (meet) of a and $a^\perp$ with respect to the given partial order, i.e., $a \wedge a^\perp$, exists in **L** for any $a \in \mathbf{L}$ and $a \wedge a^\perp = \mathbf{0}$ (law of contradiction); similarly the least upper bound (join) $a \vee a^\perp = \mathbf{1}$ for any $a \in \mathbf{L}$ (excluded middle law).

(iv) The join $\bigvee_n a_n$ exists in **L** for any sequence $\{a_n\} \subset \mathbf{L}$ of pairwise orthogonal elements of **L**, i.e., $a_n \perp a_m$ (or equivalently $a_n \le a_m^\perp$) whenever $n \ne m$ (σ-orthocompleteness condition).

(v) For any $a, b \in \mathbf{L}$, $a \le b$, one has $b = a \vee (a^\perp \wedge b) = a \vee (a \vee b^\perp)^\perp$ (orthomodular identity).

Elements of a logic represent elementary statements about some physical system. A mapping $\mathbf{s}: \mathbf{L} \to [0, 1]$ representing the state of a physical system is therefore called a state on **L** if it fulfills the following:

(S1) $\mathbf{s}(\mathbf{1}) = 1$

(S2) $\mathbf{s}(a \vee b) = \mathbf{s}(a) + \mathbf{s}(b)$ for any orthogonal $a, b \in \mathbf{L}$.

However, the nature of a physical system need not be additive. One possible way to overcome this is by replacing the ordinary addition $+$ in (S2) by a pseudoaddition $\oplus$.

Definition 1. A binary operation $\oplus$ on [0, 1] is called a pseudoaddition if it is continuous, nondecreasing in both components, associative, and $x + 0 = 0 + x = x$ for any $x \in [0, 1]$.

Note that the commutativity of a pseudoaddition follows from the above-stated axioms on $\oplus$ due to results of Ling (1965). Further, any pseudoaddition $\oplus$ is, in fact, a continuous t-conorm on [0, 1] (see, e.g., Schweizer and Sklar, 1983). The next theorem (Ling, 1965; Schweizer and Sklar, 1983) gives a full characterization of the structure of a pseudoaddition $\oplus$.

Theorem 1. Let $\oplus$ be a pseudoaddition on [0, 1]. Then there is a system of pairwise disjoint open subintervals of the unit interval $\{]\alpha_k, \beta_k[, k \in K\}$ and a system of continuous strictly increasing functions $\{g_k, k \in K\}$, $g_k: [\alpha_k, \beta_k] \to [0, +\infty]$, $g(\alpha_k) = 0$, so that for any x, $y \in [0, 1]$ one has

$$x \oplus y = \begin{cases} g_k^{-1}(\min\{g_k(x) + g_k(y); g_k(\beta_k)\}) & \text{if} \quad x, y \in]\alpha_k, \beta_k[\\ x \vee y & \text{otherwise} \end{cases}$$

Recall that $x \vee y = \sup\{x; y\}$. The system $\{\langle]\alpha_k, \beta_k[; g_k \rangle, k \in K\} \approx \oplus$ is called a representation of $\oplus$. In the following example, we present the most applied pseudoadditions (t-conorms) on [0, 1].

Example 1. (a1) *Strict pseudoaddition* $\oplus \approx \{\langle]0, 1[; g \rangle\}$, where $g(1) = +\infty$. In this case, one has $x \oplus y = g^{-1}(g(x) + g(y))$ for any $x, y \in [0, 1]$. Put, e.g., $g(x) = -\log(1 - x)$. Then

$$x \oplus y = 1 - (1 - x) \cdot (1 - y) = x + y - x \cdot y$$

i.e., $\oplus$ is the *probabilistic sum*. Note that for a given strict pseudoaddition $\oplus$, the function g is called an additive generator of $\oplus$ and it is unique up to a positive multiplicative constant.

(b1) *Nilpotent pseudoaddition* $\oplus \approx \{\langle]0, 1[; g \rangle\}$, where $g(1) = 1$. Note that it is enough to require $g(1)$ to be finite; a positive multiplicative constant (i.e., if one takes $c \cdot g$ instead g) does not change the induced $\oplus$ and hence one can always require $g(1) = 1$ in this case. Then, for a given nilpotent $\oplus$, g is unique. Take, e.g., $g(x) = x$. Then $x \oplus y = \min\{x + y, 1\}$, i.e., $\oplus$ is the bounded sum.

(c1) *The supremum* $\vee$ is a pseudoaddition with empty representation. It can be obtained, e.g., as a limit pseudoaddition of a sequence of nilpotent pseudoadditions $\{\oplus_n\}$ with generators $g_n(x) = x^n$, $n \in \mathbf{N}$.

Definition 2. Let $\oplus$ be a pseudoaddition on [0, 1] and let $\mathbf{L}$ be a logic. A mapping $\mathbf{m}: \mathbf{L} \to [0, 1]$ will be called a $\oplus$-state (a pseudoadditive state) if it fulfills the following:

(PS1) $\mathbf{m}(\mathbf{1}) = 1$ and $\mathbf{m}(0) = 0$

(PS2) $\mathbf{m}(a \vee b) = \mathbf{m}(a) \oplus \mathbf{m}(b)$ for any orthogonal $a, b \in \mathbf{L}$.

Note that (PS2) does not imply $\mathbf{m}(\mathbf{0}) = 0$. If $\mathbf{m}$ is not a constant mapping and if the only idempotents of $\oplus$ are 0 and 1, then (PS2) implies $\mathbf{m}(\mathbf{0}) = 0$. This is, e.g., the case of strict and nilpotent pseudoadditions (these are often called Archimedean t-conorms). A negative example is $_\vee$. Further, a classical state $\mathbf{s}$ is a "bounded sum" state and vice versa, any "bounded sum" state $\mathbf{m}$ is a classical state iff $\mathbf{m}(a) + \mathbf{m}(a^\perp) = 1$ for any $a \in \mathbf{L}$. There are four principal types of pseudoadditive states:

- (SS) $\oplus$-states with strict $\oplus$ (strict states).
- (NSA) $\oplus$-states with nilpotent $\oplus$ fulfilling $g(\mathbf{m}(a)) + g(\mathbf{m}(a^\perp)) = 1$ for any $a \in \mathbf{L}$ (nilpotent states additive).
- (NSP) $\oplus$-states with nilpotent $\oplus$ not included in (NSA), i.e., for some $a \in \mathbf{L}$ one has $g(\mathbf{m}(a)) + g(\mathbf{m}(a^\perp)) > 1$ (nilpotent states pseudoadditive).
- (PS) $_\vee$-states (possibility states).

A similar classification of pseudoadditive measures (with respect to an Archimedean $\oplus$) was introduced in Weber (1984). Possibility states can be defined equivalently through (PS1) and (PS2*):

(PS2*) $\mathbf{m}(a \vee b) = \mathbf{m}(a) {}_\vee\, \mathbf{m}(b)$ for any $a, b \in \mathbf{L}$.

Lemma 1. Any strict state is a transformation of a "probabilistic sum" state.

Proof. Let $\mathbf{m}$ be a $\oplus$-state where $x \oplus y = g^{-1}(g(x) + g(y))$, $g(1) = +\infty$. Put $\mathbf{m}_1 = h \circ \mathbf{m}$, where $h(x) = 1 - \exp(-g(x))$, $x \in [0, 1]$. Then $\mathbf{m}_1$ is a "probabilistic sum" state and $\mathbf{m} = h^{-1} \circ \mathbf{m}$. ■

Lemma 2. Any $\oplus$-state $\mathbf{m}$ of type (NSA) is a transformation of a classical state $\mathbf{s}$.

Proof. Let g be the generator of $\oplus$ [$\oplus$ is nilpotent and hence $g(1) = 1$]. Put $\mathbf{s} = g \circ \mathbf{m}$. Then $\mathbf{s}$ is a "bounded sum" state of type (NSA) and hence it is a classical state. Further, $\mathbf{m} = g^{-1} \circ \mathbf{s}$. ■

Lemmas 1 and 2 are special cases of the following theorem.

Theorem 2. Let $h\colon \to [0, 1]$ be a strictly increasing bijection. Let $\oplus$ be a pseudoaddition on $[0, 1]$. Then $\oplus_h$ defined via $x \oplus_h y = h^{-1}(h(x) \oplus h(y))$, for $x, y \in [0, 1]$, is a pseudoaddition on $[0, 1]$, too. Further, $\oplus_h = \oplus$ for any h iff $\oplus = {}_\vee$. Let $\mathbf{m}$ be a $\oplus$-state on a logic $\mathbf{L}$. Then $h \circ \mathbf{m}$ is a $\oplus_h$-state on $\mathbf{L}$ (of the same type as $\mathbf{m}$). If $\mathbf{m}$ is a possibility state on $\mathbf{L}$, then $h \circ \mathbf{m}$ is a possibility state on $\mathbf{L}$ for any h, too.

Example 2. Let $\mathbf{L} = (\Omega, \Delta)$ be a concrete logic and let $f\colon \Omega \to [0, 1]$ be any function such that $\sup\{f(\omega); \omega \in \Omega\} = 1$. Put $\Pi(a) = \sup_{\omega \in a} f(\omega)$ for all $a \in \mathbf{L}$. Then Π is a possibility state on $\mathbf{L}$.

Lemma 3. Let $\mathbf{L}$ be an atomic logic. A mapping $\Pi: \mathbf{L} \to [0, 1]$ is a possibility state on $\mathbf{L}$ if and only if there is a mapping $f: A \to [0, 1]$, $\sup\{f(a); a \in A\} = 1$, where A is the system of all atoms of $\mathbf{L}$.

Proof. It is enough to put $f(a) = \Pi(a)$, $a \in A$. Vice versa, for any $b \in \mathbf{L}$, one has $\Pi(b) = \sup\{f(a); a \leq b, a \in A\}$. ■

Remark 1. If $\mathbf{m}$ is a two-valued pseudoadditive state on $\mathbf{L}$, then it is a $\oplus$-state for any pseudoaddition $\oplus$, and vice versa. If $\mathbf{m}$ is an (NSA)-type pseudoadditive state for some nilpotent pseudoaddition, then it is two-valued iff it is both a classical state on $\mathbf{L}$ and a possibility state on $\mathbf{L}$.

3. $\oplus$-STATES ON A PTÁK SUM

Let $\mathbf{B}$ be a Boolean algebra and let $\mathbf{L}$ be a logic. The Pták (1986) sum $\mathscr{L}$ of $\mathbf{B}$ and $\mathbf{L}$, $\mathscr{L} = \mathbf{B} + \mathbf{L}$, is a logic which may be viewed as a system of all possible n-tuples $p = ((a_1, b_1), \ldots, (a_n, b_n))$, $n \in \mathbf{N}$, where $a_i \in \mathbf{B}$ and $b_i \in \mathbf{L}$, $i = 1, 2, \ldots, n$, $a_1 \vee \cdots \vee a_n = \mathbf{1_B}$, $a_1 \perp a_j$ whenever $i \neq j$. Recall that for $r = ((c_1, d_1), \ldots (c_m, d_m)) \in \mathscr{L}$, one has

$$r \leq p \quad \text{iff} \quad d_i \leq b_j \quad \text{whenever} \quad c_i \wedge a_j \neq \mathbf{0_B}$$

$$r \perp p \quad \text{iff} \quad d_i \perp b_j \quad \text{whenever} \quad c_i \wedge a_j \neq \mathbf{0_B}$$

$$r \vee p = ((a_j \wedge c_i, b_j \vee d_i), \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n)$$

The embeddings f_1 and f_2, respectively, of $\mathbf{B}$ and $\mathbf{L}$, respectively, into $\mathscr{L}$ are the following:

$$f_1(a) = ((a, \mathbf{1_L}), (a^{\perp}, \mathbf{0_L})) \qquad \text{for} \quad a \in \mathbf{B}$$

$$f_2(b) = ((\mathbf{1_B}, b)) \qquad \text{for} \quad b \in \mathbf{L}$$

For more details see Pták (1986) or Janiš and Riečanová (1992). Let $\mathbf{s}_1, \mathbf{s}_2$, respectively, be states on $\mathbf{B}, \mathbf{L}$, respectively. For any element $p \in \mathscr{L}$, put

$$\mathbf{s}(p) = \sum_{i=1}^{n} \mathbf{s}_1(a_i) \cdot \mathbf{s}_2(b_i)$$

Then $\mathbf{s}$ is a state on $\mathscr{L}$ and $\mathbf{s}_i = \mathbf{m} \circ f_i$, $i = 1, 2$.

In the following, we will extend the foregoing results on the Pták sum and the states to the case of pseudoadditive states. For a given pseudoaddition $\oplus$, let $\mathbf{m}_1$ be a $\oplus$-state on $\mathbf{B}$ and let $\mathbf{m}_2$ be a $\oplus$-state on $\mathbf{L}$. Under which conditions is there a $\oplus$-state $\mathbf{m}$ on $\mathscr{L}$ so that $\mathbf{m}_i = \mathbf{m} \circ f_i$, $i = 1, 2$? For this purpose we have to look for a pseudomultiplication $\odot$ on $[0, 1]$ with some "convenient" properties. Then we will expect $\mathbf{m}$ in the following form: for a $p \in \mathscr{L}$, one has

$$\mathbf{m}(p) = (\mathbf{m}_1(a_1) \odot \mathbf{m}_2(b_1)) \oplus \cdots \oplus (\mathbf{m}_1(a_n) \odot \mathbf{m}_2(b_n))$$

Let us denote $\mathbf{I}_1 = \{\mathbf{m}_1(a); a \in \mathbf{B}\}$ and $\mathbf{I}_2 = \{\mathbf{m}_2(b); b \in \mathbf{L}\}$. Then $\mathbf{m}$ can be a $\oplus$-state only if $\odot$ fulfills the following:

(1) $x \odot 1 = x$ for any $x \in I_1$ and $1 \odot z = z$ for any $z \in I_2$.
(2) $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ and $x \odot (w \oplus z) = (x \odot w) \oplus (x \odot z)$ for any $x, y \in \mathbf{I}_1$ and $z, w \in \mathbf{I}_2$ such that $x = \mathbf{m}_1(a_x)$ and $y = \mathbf{m}_1(a_y)$, $a_x \perp a_y$, and $z = \mathbf{m}_2(b_z)$, $w = \mathbf{m}_2(b_2)$, $b_z \perp b_w$.
(3) $x \odot z = 0$ for $x \in \mathbf{I}_1$, $z \in \mathbf{I}_2$ iff $x = 0$ or $z = 0$.

If $\mathbf{m}$ is a pseudoadditive state of type (SS), (NSP), or (PS), any x, y, z, w from the open unit interval may occur in (2) in general. However, if $\mathbf{m}$ is of type (NSA), the situation is rather different: let g be a generator of $\oplus$; then only x, y, z, w fulfilling

$$g(x) + g(y) \leq 1 \qquad \text{and} \qquad g(z) + g(w) \leq 1$$

may occur in (2). These facts together with some other natural requirements lead to the following definition.

Definition 3. Let $\oplus$ be a pseudoaddition on [0, 1]. A binary operation $\odot$ on [0, 1] will be called a pseudomultiplication corresponding (A-corresponding) to $\oplus$ if it satisfies the following:

(M1) 1 is both the left and the right unit, i.e., $x \odot 1 = x$ and $1 \odot z = z$ for any $x, z \in [0, 1]$.
(M2) $\oplus$ is distributive with respect to $\odot$, i.e.,

$$(x \oplus y) \odot (z \oplus w) = (x \odot z) \oplus (x \odot w) \oplus (y \odot z) \oplus (y \odot w)$$
$$\text{for any } x, y, z, w \in [0, 1]$$

(M3) $x \odot z = 0$ iff $x = 0$ or $x = 0$.
(M4) $\odot$ is nondecreasing in both components.
(M5) $\odot$ is continuous.

The A-correspondence of $\odot$ to $\oplus$ is defined only for nilpotent pseudoadditions $\oplus$ (with a generator g). We replace only (M2) by (M2A), where the restricted distributivity is required, i.e., we deal only with x, y, z, w from the unit interval satisfying $g(x) + g(y) \leq 1$ and $g(z) + g(w) \leq 1$.

Theorem 3. Let $\oplus$ be a pseudoaddition and let $\odot$ be a pseudomultiplication corresponding to $\oplus$ (A-corresponding to $\oplus$). Let $\mathbf{B}$ be a Boolean algebra and let $\mathbf{L}$ be a logic. Let $\mathbf{m}_1$ be a $\oplus$-state on $\mathbf{B}$ of some type and let $\mathbf{m}_2$ be a $\oplus$-state on $\mathbf{L}$ of the same type as $\mathbf{m}_1$. then

$$\mathbf{m}(p) = (\mathbf{m}_1(a_1) \odot \mathbf{m}_2(b_1)) \oplus \cdots \oplus (\mathbf{m}_1(a_n) \odot (\mathbf{m}_2(b_n)), \qquad p \in \mathscr{L}$$

defines a $\oplus$-state on $\mathscr{L}$ (of the same type as $\mathbf{m}_1$ and $\mathbf{m}_2$) such that $\mathbf{m}_i = \mathbf{m} \circ f_i$, $i = 1, 2$.

The proof of the previous theorem is an easy consequence of Definition 3. Note only that in the case of $\oplus$-states of type (NSA) we have to deal with an A-corresponding pseudomultiplication $\odot$.

Lemma 4. Let $\oplus$ be a strict or a nilpotent pseudoaddition. Then there is no pseudomultiplication $\odot$ corresponding to $\oplus$.

Proof. Let $\odot$ be a pseudomultiplication corresponding to $\oplus$. Then for any $z \in]0, 1[$, $x_n = 1 - 1/n$, $n = 1, 2, \ldots$, one has $\lim x_n \odot z = 1 \odot z = z$. Further, $x_n \oplus x_n < 1$ implies

$$(x_n \oplus x_n) \odot z = (x_n \odot z) \oplus (x_n \odot z) \leq 1 \odot z = z$$

and consequently $z \oplus z \leq z$. But this is a contradiction with the Archimedean property of $\oplus$ claiming $z \oplus z > z$ for any nontrivial z. ■

Example 3. Let $\oplus$ be a nilpotent pseudoaddition with generator g. Put

$$x \odot z = g^{-1}(g(x) \cdot g(z))$$

for any $x, z \in [0, 1]$. Then $\odot$ is a pseudomultiplication A-corresponding to $\oplus$.

Remark 2. Let $\oplus$ be a nilpotent pseudoaddition with generator g. Let $\mathbf{B}$ be a Boolean algebra and let $\mathbf{m}_1$ be an (NSA)-type $\oplus$-state on $\mathbf{B}$. Let $\mathbf{L}$ be a logic and let $\mathbf{m}_2$ be an (NSA)-type $\oplus$-state on $\mathbf{L}$. Let $\odot$ be a pseudomultiplication introduced in Example 3. By Theorem 2, there is a $\oplus$-state $\mathbf{m}$ on the Pták sum $\mathscr{L}$ induced by $\mathbf{m}_1$ and $\mathbf{m}_2$. We get

$$\mathbf{m}(p) = g^{-1}\left(\sum_{i=1}^{n} g(\mathbf{m}_1(a_1)) \cdot g(\mathbf{m}_2(b_1)) \right) \qquad \text{for} \quad p \in \mathscr{L}$$

Following Lemma 2, we see that this situation corresponds (up to the transformation g) to the situation dealing with classical states.

Example 4. Let h, q be any strict increasing continuous bijections from the unit interval into the unit interval such that $h(x) \leq x$, $q(x) \leq x$ for any $x \in [0, 1]$. Put

$$x \underset{1}{\odot} z = \max\{h(x) \cdot z; x \cdot q(z)\}$$

$$x \underset{2}{\odot} z = \max\{\min\{h(x), z\}; \min\{x, q(z)\}\}$$

for any $x, z \in [0, 1]$. Then both $\odot_1$ and $\odot_2$ are pseudomultiplications corresponding to $_\vee$ which may not be commutative.

Remark 3. Let $h = q$ in Example 4 be the identity on [0, 1], i.e.,

$$x \underset{1}{\odot} z = x \cdot z \qquad \text{and} \qquad x \underset{2}{\odot} z = \min\{x, z\}$$

Let $\mathbf{m}_1$ be a possibility state on a Boolean algebra $\mathbf{B}$ and let $\mathbf{m}_2$ be a possibility state on a logic $\mathbf{L}$. Let us define, for any $p \in \mathscr{L} = \mathbf{B} + \mathbf{L}$,

$$\mathbf{m}(p) = \max\{\mathbf{m}_1(a_i) \cdot \mathbf{m}_2(b_i); i = 1, 2, \ldots, n\}$$

$$\mathbf{M}(p) = \max\{\min\{\mathbf{m}_1(a_i), \mathbf{m}_2(b_i)\}; i = 1, 2, \ldots, n\}$$

Then both $\mathbf{m}$ and $\mathbf{M}$ are possibility states on $\mathscr{L}$ such that $\mathbf{m} \circ f_i = \mathbf{M} \circ f_i = \mathbf{m}_i$, $i = 1, 2$. Note that $\mathbf{m}$ is similar to the Shilkret (1971) integral and $\mathbf{M}$ to the Sugeno (1974) integral for possibility measures.

REFERENCES

Beltrametti, E. G., and Cassinelli, G. (1981). *The Logic of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Butnariu, D., and Klement, E. P. (1991). Triangular norm-based measures and their Markov-kernel representation, *Journal of Mathematical Analysis and Its Applications*, **162**, 111–143.

Janiš, V., and Riečanová, Z. (1992). Completeness in sums of Boolean algebras and logics, *International Journal of Theoretical Physics*, **31**(9), 119–127.

Klement, E. P., and Weber, S. (1992). Generalized measures, *Fuzzy Sets and Systems*, **40**, 375–394.

Ling, C. H. (1965). Representation of associative functions, *Publicationes Mathematicae Debrecen*, **12**, 189–212.

Pták, P. (1986). Summing of Boolean algebras and logics, *Demonstratio Mathematica*, **19**(2), 349–357.

Riečanová, Z. (1982). About σ-additive and σ-maxitive measures, *Mathematica Slovaca*, **32**(4), 389–395.

Schweizer, B., and Sklar, A. (1983). *Probabilistic Metric Spaces*, North Holland, Amsterdam.

Shilkret, N. (1971). Maxitive measure and integration, *Indagationes Mathematicae*, **33**, 109–116.

Sugeno, M. (1974). Theory of fuzzy integrals and its applications. Ph.D. Thesis, Tokyo Institute of Technology.

Sugeno, M., and Murofushi, T. (1987). Pseudo-additive measures and integrals, *Journal of Mathematical Analysis and Its Applications*, **122**, 197–222.

Varadarajan, V. S. (1968). *Geometry of Quantum Theory*, Van Nostrand, New York.

Weber, S. (1984). $\perp$-decomposable measures and integrals for Archimedean t-conorms $\perp$, *Journal of Mathematical Analysis and Its Applications*, **101**, 114–138.

Zadeh, L. A. (1978). Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, **1**, 321–327.